Partial Fraction

1. $-\dfrac{2x+1}{2(x^2+1)}+\dfrac{1}{x-1}-\dfrac{1}{2(x-1)^2}$

2. $A=\dfrac{g(a)}{(a-b)(a-c)}$, $B=\dfrac{g(b)}{(b-a)(b-c)}$, $C=\dfrac{g(c)}{(c-a)(c-b)}$, $\dfrac{1/2}{x-1}-\dfrac{4}{x-2}+\dfrac{9/2}{x-3}$

3. L.H.S. $=-\dfrac{a^3(b-c)+b^3(c-a)+c^3(a-b)}{abc(a-b)(b-c)(c-a)}$ (1)

Let $f(a,b,c) = a^3(b-c)+b^3(c-a)+c^3(a-b)$

$f(b,b,c) = b^3(b-c) + b^3(c-b) + c^3(b-b) = 0$

By factor theorem, $(a-b)$ is a factor of f.

Since f is cyclic, $(a-b)(b-c)(c-a)$ is a factor of f.

Since $\deg\{f\} = 4$, $\deg\{(a-b)(b-c)(c-a)\} = 3$.

$f(a,b,c) = k(a+b+c)(a-b)(b-c)(c-a)$ (2)

By comparing the coefficients of a^3b – term on both sides of (2), we get $k = -1$

$\therefore$ $f(a,b,c) = -(a+b+c)(a-b)(b-c)(c-a)$ (3)

From (1) and (3), L.H.S. $=-\dfrac{-(a+b+c)(b-c)(c-a)(a-b)}{abc(a-b)(b-c)(c-a)}=\dfrac{(a+b+c)}{abc}=$ R.H.S.

4. $\dfrac{3}{13(2x^2+5)}-\dfrac{5}{52(x+2)}+\dfrac{5}{52(x-2)}$

5. Let $f(x)=\dfrac{a^2(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{b^2(x-c)(x-a)}{(b-c)(b-a)}+\dfrac{c^2(x-a)(x-b)}{(c-a)(c-b)}-x^2$

Then $f(a) = f(b) = f(c) = 0$

But $\deg\{f\} = 2$ and $f(x) = 0$ has 3 roots $x = a, b, c$.

By the Identity Theorem, $f(x) \equiv 0$. Result follows.

6. **(a)** By division,

$2x^3-8x^2+x+6 \equiv 2(x-3)^3+10(x-3)^2+7(x-3)-9$

$\dfrac{2x^3-8x^2+x+6}{(x-3)^4}=\dfrac{2}{x-3}+\dfrac{10}{(x-3)^2}+\dfrac{7}{(x-3)^3}-\dfrac{9}{(x-3)^4}$

3	2	-8	+1	+6
		6	-6	-15
3	2	-2	-5	**-9**
		6	+12	
3	2	+4	**+7**	
		6		
	2	**+10**		

(b) $\dfrac{1}{x-3}-\dfrac{x+2}{x^2-x+1}-\dfrac{2x+3}{(x^2-x+1)^2}$

7. $(x-a)^4 = [(x+a)-2a]^4$

$= (x+a)^4 - 4(x+a)^3(2a) + 6(x+a)(2a)^2 - 4(x+a)(2a)^3 + (2a)^4$

$= (x+a)^4 - 8a(x+a)^3 + 24a^2(x+a)^2 - 32a^2(x+a) + 16a^4$

$\dfrac{(x-a)^4}{(x+a)^4}=1-\dfrac{8a}{x+a}+\dfrac{24a^2}{(x+a)^2}-\dfrac{32a^3}{(x+a)^3}+\dfrac{16a^4}{(x+a)^4}$

8. $y=\dfrac{a-b}{(x-a)(x-b)}=\dfrac{1}{x-a}-\dfrac{1}{x-b}$

$y^2=\left(\dfrac{1}{x-a}-\dfrac{1}{x-b}\right)^2=\left(\dfrac{1}{x-a}\right)^2-\dfrac{2}{(x-a)(x-b)}+\left(\dfrac{1}{x-b}\right)^2=\left(\dfrac{1}{x-a}\right)^2-\dfrac{2/(a-b)}{x-a}+\dfrac{2/(a-b)}{x-b}+\left(\dfrac{1}{x-b}\right)^2$

9. **(a)** $3-\frac{14}{x-1}+\frac{27}{x-2}$ **(b)** $-\frac{2}{9(1+2x)}-\frac{1}{9(1-x)}+\frac{4}{3(1-x)^2}$ **(c)** $\frac{1}{4(1-x)}+\frac{1}{4(1+x)}-\frac{1}{2(1+x)^2}$

(d) $\frac{4}{1-2x}+\frac{2}{1+x}+\frac{3}{(1+x)^2}$ **(e)** $\frac{2}{x+1}-\frac{1}{x-1}-\frac{2}{2x-1}$ **(f)** $-1+\frac{1}{x+3}-\frac{3}{x+2}-\frac{2}{x-2}$

(g) $\frac{1}{2(1+x^2)}-\frac{1}{2(1+x)^2}$

10. $\frac{1/2}{x}-\frac{1/2}{x+2}$, S(n) = $\frac{1}{2}\left[\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right]=\frac{n(3n+5)}{4(n+1)(n+2)}$

11. Put y = x – 1

$$\frac{1}{x(x-2)(x-1)^{2n}}=\frac{1}{(y+1)(y-1)y^{2n}}=\frac{1}{(y^2-1)y^{2n}}=-\frac{1}{(1-y^2)y^{2n}}$$

$$=-\frac{1}{y^{2n}}\left[1+y^2+y^4+\ldots+y^{2n-2}+y^{2n}+y^{2n+2}+y^{2n+4}+\ldots\right]=-\frac{1}{y^{2n}}-\frac{1}{y^{2n-2}}+\ldots+-\frac{1}{y^2}-\left(1+y^2+y^4+\ldots\right)$$

$$=-\frac{1}{y^{2n}}-\frac{1}{y^{2n-2}}+\ldots+-\frac{1}{y^2}-\frac{1}{1-y^2}=-\frac{1}{y^{2n}}-\frac{1}{y^{2n-2}}-\ldots-\frac{1}{y^2}-\frac{1/2}{1-y}-\frac{1/2}{1+y}$$

$$=-\frac{1}{(x-1)^{2n}}-\frac{1}{(x-1)^{2n-2}}-\ldots-\frac{1}{(x-1)^2}-\frac{1/2}{x-2}-\frac{1/2}{x}$$

12. $\frac{3}{2x}-\frac{1}{x+1}-\frac{1}{2(x+2)}$

13. $\frac{x}{(x+1)(x+2)(x+3)}=\frac{-1/2}{x+1}+\frac{2}{x+2}+\frac{-3/2}{x+3}$

$$\sum_{n=1}^{\infty}\frac{n}{(n+1)(n+2)(n+3)}=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{-1/2}{i+1}+\frac{2}{i+2}+\frac{-3/2}{i+3}\right)$$

$$=\lim_{n\to\infty}\left(\frac{-1/2}{2}+\frac{2}{3}+\frac{-1/2}{3}+\frac{-3/2}{n+2}+\frac{2}{n+1}+\frac{-3/2}{n+1}\right)=\frac{1}{4}$$

14. **(a)** $\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}=\frac{1}{2}\sum_{r=1}^{n}\left(\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right)=\frac{1}{2}\left(\frac{1}{1\times 2}-\frac{1}{(n+1)(n+2)}\right)=\frac{n(n+3)}{4(n+1)(n+2)}$

(b) $\sum_{r=1}^{n}\frac{3r+1}{(r+1)(r+2)(r+3)}=\sum_{r=1}^{n}\frac{3(r+1)-2}{(r+1)(r+2)(r+3)}=\sum_{r=1}^{n}\left[\frac{3}{(r+2)(r+3)}-\frac{2}{(r+1)(r+2)(r+3)}\right]$

$=3\sum_{r=1}^{n}\left[\frac{1}{r+1}-\frac{1}{r+3}\right]-2\frac{(n+1)(n+4)}{4(n+2)(n+3)}$, by (a)

$=3\left[\frac{1}{3}-\frac{1}{n+3}\right]-\frac{(n+1)(n+4)}{2(n+2)(n+3)}=\frac{n(5n+7)}{6(n+2)(n+3)}$

15. **(a)** $(1+x)^m=\sum_{r=0}^{m}C_r^m x^r$. Put $x=1$, $\sum_{r=0}^{m}C_r^m=(1+1)^m=2^m$.

(b) From (a), $\sum_{r=0}^{2n+1}C_r^{2n+1}=2^{2n+1}$ …..(1)

Since $C_{n+r}^{2n+1}=C_{(2n+1)-(n+r)}^{2n+1}=C_{n-r+1}^{2n+1}$, r = 1, 2, …, n

$$\therefore \quad \sum_{r=0}^{2n+1} C_r^{2n+1} = 2\sum_{r=0}^{n} C_r^{2n+1} \qquad .....(2)$$

From (1) and (2), $\sum_{r=0}^{n} C_r^{2n+1} = \frac{2^{2n+1}}{2} = 2^{2n}$

i.e. $C_1^{2n+1} + C_2^{2n+1} + ... + C_n^{2n+1} = 2^{2n} - 1 \qquad .....(3)$

Divide both sides by $(2n + 1)!$ on both sides of (3), result follows.

16. Since the degree of the numerator is equal to that of the denominator and both are monic, we can write:

$$\frac{(x-1)(x-2)\cdots(x-n)}{(x+1)(x+2)\cdots(x+n)} = 1 + \frac{A_1}{x+1} + \frac{A_2}{x+2} + ... + \frac{A_n}{x+n}$$, where A_i are constants.

$$(x-1)(x-2)\cdots(x-n) = (x+1)(x+2)\cdots(x+n) + A_1(x+2)\cdots(x+n) + ... + A_r(x+1)...(x+r-1)(x+r+1)...(x+n) + ... + A_n(x+1)(x+2)\cdots(x+n-1)$$

By putting $x = -r$,

$(-r-1)(-r-2)\ldots(-r-n) = A_r(-r+1)(-r+2)\ldots(-1)(1)(2)\ldots(-r+n)$

$$A_r = \frac{(-1)^{n-r+1}(n+r)!}{(n-r)!r!(r-1)!(x+r)}$$ and we have $$\frac{(x-1)(x-2)\cdots(x-n)}{(x+1)(x+2)\cdots(x+n)} = 1 + \sum_{r=1}^{n} \frac{(-1)^{n-r+1}(n+r)!}{(n-r)!r!(r-1)!(x+r)}.$$

Putting $x = 0$ in the above identity, $$\frac{(-1)(-2)\cdots(-n)}{(1)(2)\cdots(n)} = 1 + \sum_{r=1}^{n} \frac{(-1)^{n-r+1}(n+r)!}{(n-r)!r!(r-1)!(r)}$$

$$(-1)^n = 1 + \sum_{r=1}^{n} \frac{(-1)^{2r}(-1)^{n-r+1}(n+r)!}{(n-r)!r!(r-1)!(r)} = 1 + (-1)^n \sum_{r=1}^{n} \frac{(-1)^{r+1}(n+r)!}{(n-r)!r!(r-1)!(r)}$$

$$\sum_{r=1}^{n} \frac{(-1)^{r+1}(n+r)!}{(n-r)!r!(r-1)!(r)} = \frac{(-1)^n - 1}{(-1)^n} = 1 - (-1)^{-n} = 1 - (-1)^{-n}(-1)^{2n} = 1 - (-1)^n$$

17. $$\frac{1}{x(x+1)\cdots(x+n)} = \sum_{r=0}^{n} \frac{C_r}{x+r}$$

$\therefore$ $C_0(x + 1)\ldots(x + n) + C_1 x(x + 2)\ldots(x + n) + \ldots + C_r x(x + 1)\ldots(x + r - 1)(x + r + 1)\ldots(x + n) + \ldots +$ $C_n x(x + 1)\ldots(x + n - 1) \equiv 1$

Put $x = -r$, $C_r(-r)[(-r) + 1]\ldots[(-r) + r - 1][(-r) + r + 1]\ldots[(-r) + n] \equiv 1.$

$\therefore$ $C_r = (-1)^r \frac{1}{r!(n-r)!}$ and $$\frac{1}{x(x+1)\cdots(x+n)} = \sum_{r=0}^{n} (-1)^r \frac{1}{r!(n-r)!(x+r)}$$

Replace n by 0, $\frac{1}{x} = \frac{1}{x}$

Replace n by 1, $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$

Replace n by 2, $\frac{1}{x(x+1)(x+2)} = \frac{1}{2!x} - \frac{1}{x+1} + \frac{1}{2!(x+2)}$

…..

Replace n by n, $$\frac{1}{x(x+1)(x+2)...(x+n)} = \frac{1}{n!x} - \frac{1}{(n-1)!(x+1)} + \frac{1}{2!(n-2)!(x+2)} + ... + \frac{(-1)^n}{n!(x+n)}$$

$$\sum_{r=0}^{n} \frac{a^r}{x(x+1)\cdots(x+r)} = \frac{1}{x} + \frac{a}{x(x+1)} + \frac{a^2}{x(x+1)(x+2)} + ... + \frac{a^n}{x(x+1)...(x+n)}$$

$$=\frac{1}{x}+\left(\frac{a}{x}-\frac{a}{x+1}\right)+\left(\frac{a^2}{2!x}-\frac{a^2}{x+1}+\frac{a^2}{2!(x+2)}\right)+\ldots+\left(\frac{a^n}{n!x}-\frac{a^n}{(n-1)!(x+1)}+\frac{a^n}{2!(n-2)!(x+2)}+\ldots+\frac{(-1)^n a^n}{n!(x+n)}\right)$$

$$=\left(\frac{1}{0!}+\frac{a}{1!}+\frac{a^2}{2!}+\ldots+\frac{a^n}{n!}\right)\frac{1}{x}-\left(\frac{a}{0!}+\frac{a^2}{1!}+\ldots+\frac{a^n}{(n-1)!}\right)\frac{1}{x+1}+\left(\frac{a^2}{2!}+\frac{a^3}{2!1!}+\ldots+\frac{a^n}{2!(n-2)!}\right)\frac{1}{x+2}+\ldots+\frac{(-1)^n a^n}{n!(x+n)}$$

$$=\frac{A_0}{n!x}-\frac{A_1}{(n-1)!(x+1)}+\frac{A_2}{2!(N-2)!(x+2)}+\ldots+\frac{(-1)^n A_n}{n!(x+n)},$$

$$=\sum_{s=0}^{n}\frac{(-1)^s A_s}{n!(n-s)!(x+s)}$$ where the A_s are polynomials of degree n in a with integral coefficients.

18. (Complex number theory is needed)

For $f(x) = x^5 - 1 = 0$, $x = 1^{\frac{1}{5}} = (\text{cis } 2n\pi)^{\frac{1}{5}} = \text{cis}\frac{2n\pi}{5}$, $n = 0, 1, 2, 3, 4$, by de Moivres' Theorem.

$$\therefore\ f(x) = (x-1)\left[x-\text{cis}\frac{2\pi}{5}\right]\left[x-\text{cis}\frac{4\pi}{5}\right]\left[x-\text{cis}\frac{6\pi}{5}\right]\left[x-\text{cis}\frac{8\pi}{5}\right]$$

$f'(x) = 5x^4$.

By extended partial fraction theorem,

$$\frac{x}{x^5-1}=\frac{1}{5(x-1)}+\frac{\text{cis}\frac{2\pi}{5}}{5\left(\text{cis}\frac{2\pi}{5}\right)^4\left(x-\text{cis}\frac{2\pi}{5}\right)}+\frac{\text{cis}\frac{4\pi}{5}}{5\left(\text{cis}\frac{4\pi}{5}\right)^4\left(x-\text{cis}\frac{4\pi}{5}\right)}+\frac{\text{cis}\frac{6\pi}{5}}{5\left(\text{cis}\frac{6\pi}{5}\right)^4\left(x-\text{cis}\frac{6\pi}{5}\right)}+\frac{\text{cis}\frac{8\pi}{5}}{5\left(\text{cis}\frac{8\pi}{5}\right)^4\left(x-\text{cis}\frac{8\pi}{5}\right)}$$

$$=\frac{1}{5}\left[\frac{1}{x-1}+\frac{\text{cis}\frac{2\pi}{5}}{x\text{cis}\frac{8\pi}{5}-1}+\frac{\text{cis}\frac{4\pi}{5}}{x\text{cis}\frac{6\pi}{5}-1}+\frac{\text{cis}\frac{6\pi}{5}}{x\text{cis}\frac{4\pi}{5}-1}+\frac{\text{cis}\frac{8\pi}{5}}{x\text{cis}\frac{2\pi}{5}-1}\right]$$

$$=\frac{1}{5}\left[\frac{1}{x-1}+\frac{\text{cis}\frac{2\pi}{5}\left(x\text{cis}\frac{2\pi}{5}-1\right)+\text{cis}\frac{8\pi}{5}\left(x\text{cis}\frac{8\pi}{5}-1\right)}{\left(x\text{cis}\frac{8\pi}{5}-1\right)\left(x\text{cis}\frac{2\pi}{5}-1\right)}+\frac{\text{cis}\frac{4\pi}{5}\left(x\text{cis}\frac{4\pi}{5}-1\right)+\text{cis}\frac{6\pi}{5}\left(x\text{cis}\frac{6\pi}{5}-1\right)}{\left(x\text{cis}\frac{6\pi}{5}-1\right)\left(x\text{cis}\frac{4\pi}{5}-1\right)}\right]$$

$$=\frac{1}{5}\cdot\frac{1}{x-1}+\frac{2}{5}\cdot\frac{x\cos\frac{4\pi}{5}-2\cos\frac{2\pi}{5}}{x^2-2x\cos\frac{2\pi}{5}+1}+\frac{2}{5}\cdot\frac{x\cos\frac{8\pi}{5}-2\cos\frac{4\pi}{5}}{x^2-2x\cos\frac{4\pi}{5}+1}$$

Note that $\text{cis}\frac{4\pi}{5}+\text{cis}\frac{6\pi}{5}=\text{cis}\frac{4\pi}{5}+\text{cis}\left(-\frac{4\pi}{5}\right)=\text{cis}\frac{4\pi}{5}+\overline{\text{cis}\frac{4\pi}{5}}=2\cos\frac{4\pi}{5}$.

19. (a) For $1 \le p \le n-1$

$$\frac{x^p}{(x+1)\cdots(x+n)}=\sum_{r=1}^{n}\frac{A_r}{x+r}$$

$x^p = A_1 (x + 2) \ldots (x + n) + A_2 (x + 1)(x + 3) \ldots (x + n) + \ldots + A_n(x + 1) \ldots (x + n - 1)$

Put $x = -r$, $(r = 1, 2, \ldots, n)$

$(-r)^p = A_r(-r+1)(-r+2)\ldots(-r+r-1)(-r+r+1)\ldots(-r+n)$

$(-1)^p r^p = A_r (-1)^{r-1} (r-1)!\,(n-r)!$

$$\therefore \quad A_r = \frac{(-1)^{p-r+1} r^p}{(r-1)!(n-r)!}, \quad (r = 1, 2, \ldots, n)$$

$$\therefore \quad \frac{x^p}{(x+1)\cdots(x+n)} = \sum_{r=1}^{n} \frac{(-1)^{p-r+1} r^p}{(r-1)!(n-r)!(x+r)}$$

(b) For p = n,

$$\frac{x^n}{(x+1)\cdots(x+n)} = 1 + \sum_{r=1}^{n} \frac{A_r}{x+r}$$

$x^n = (x + 1)(x + 2) \ldots (x + n) + A_1 (x + 2) \ldots (x + n) + A_2 (x + 1)(x + 3) \ldots (x + n) + \ldots +$ $A_n(x + 1) \ldots (x + n - 1)$

Put $x = -r$, $(r = 1, 2, \ldots, n)$

$$\therefore \quad A_r = \frac{(-1)^{n-r+1} r^n}{(r-1)!(n-r)!}, \quad (r = 1, 2, \ldots, n)$$

$$\therefore \quad \frac{x^n}{(x+1)\cdots(x+n)} = 1 + \sum_{r=1}^{n} \frac{(-1)^{n-r+1} r^n}{(r-1)!(n-r)!(x+r)}$$

Put $x = 0$ in (a),

$$0 = \sum_{r=1}^{n} \frac{(-1)^{p-r+1} r^p}{(r-1)!(n-r)!r} = (-1)^p \sum_{r=1}^{n} \frac{(-1)^{r-1} r^p}{(r-1)!(n-r)!r}, \quad \text{since} \quad (-1)^{-r+1} = (-1)^{r-1}.$$

$$\therefore \quad \sum_{r=1}^{n} \frac{(-1)^{r-1} r^p}{r!(n-r)!} = 0, \quad \text{for} \quad p = 0, 1, 2, \ldots, (n-1).$$

Put $x = 0$ in (b),

$$0 = 1 + \sum_{r=1}^{n} \frac{(-1)^{n-r+1} r^n}{(r-1)!(n-r)!r} = 1 + (-1)^n \sum_{r=1}^{n} \frac{(-1)^{r-1} r^n}{(r-1)!(n-r)!r}, \quad \text{since} \quad (-1)^{-r+1} = (-1)^{r-1}.$$

$$\therefore \quad \sum_{r=1}^{n} \frac{(-1)^{r-1} r^n}{(r-1)!(n-r)!r} = (-1)^{n-1}, \quad \text{since} \quad (-1)^{-n+1} = (-1)^{n-1}.$$

20. $$\frac{n!}{y(y+1)(y+2)....(y+n)} = \sum_{r=0}^{n} \frac{A_r}{y+r}$$

$\therefore$ $n! = A_0(y + 1) \ldots(y + n) + A_1 y(y + 2)\ldots (y + n) + \ldots + A_n y(y + 1) \ldots (y + n - 1)$

Put $y = -r$, $(r = 0, 1, 2, \ldots, n)$

$\therefore$ $n! = A_r (-r)(-r + 1) \ldots (-2)(-1)(1)(2) \ldots (-r + n) = A_r (-1)^r r! (n - r)!$

$$\therefore \quad A_r = \frac{(-1)^r n!}{r!(n-r)!} = (-1)^r c_r, \quad \text{since} \quad (-1)^{-r} = (-1)^r.$$

$$\therefore \quad \frac{n!}{y(y+1)(y+2)....(y+n)} = \sum_{r=0}^{n} \frac{A_r}{y+r} = \sum_{r=0}^{n} \frac{(-1)^r c_r}{y+r}$$

21. **(a)** $f(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$ and $a_1, a_2, \ldots, a_n$ are unequal.

$\ln f(x) = \ln(x - a_1) + \ln(x - a_2) + \ldots + \ln(x - a_n)$

Differentiate with respect to x on both sides,

$$\therefore \quad \frac{f'(x)}{f(x)} = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \ldots + \frac{1}{x-a_n} = \sum_{r=1}^{n} \frac{1}{x-a_r}$$

(b) Differentiate the result of (a) with respect to x again,

$$\frac{f(x)f''(x) - [f'(x)]^2}{[f(x)]^2} = \sum_{r=1}^{n} \frac{-1}{(x-a_r)^2}$$

$$\therefore \quad \frac{[f'(x)]^2 - f(x)f''(x)}{[f(x)]^2} = \sum_{r=1}^{n} \frac{1}{(x-a_r)^2}$$

(c) $\frac{\phi(x)}{f(x)} = \sum_{r=1}^{n} \frac{A_r}{x-a_r}$, where A_r are constants.

$$\phi(x) = \sum_{r=1}^{n} \frac{A_r f(x)}{x-a_r} = \sum_{r=1}^{n} A_r (x-a_1)\ldots(x-a_{r-1})(x-a_{r+1})\ldots(x-a_n)$$

Put $x = a_r$, $\phi(a_r) = A_r(a_r - a_1)\ldots(a_r - a_{r-1})(a_r - a_{r+1})\ldots(a_r - a_n)$ ….(1)

From (a), $\frac{f'(x)}{f(x)} = \sum_{r=1}^{n} \frac{1}{x-a_r}$

$$f'(x) = \sum_{r=1}^{n} \frac{f(x)}{x-a_r} = \sum_{r=1}^{n} (x-a_1)\ldots(x-a_{r-1})(x-a_{r+1})\ldots(x-a_n)$$

Put $x = a_r$, $f'(a_r) = (a_r - a_1)\ldots(a_r - a_{r-1})(a_r - a_{r+1})\ldots(a_r - a_n)$ ….(2)

Compare (1) and (2), $A_r = \frac{\phi(a_r)}{f'(a_r)}$

$$\therefore \quad \frac{\phi(x)}{f(x)} = \sum_{r=1}^{n} \frac{A_r}{x-a_r} = \sum_{r=1}^{n} \frac{\phi(a_r)}{f'(a_r)} \frac{1}{x-a_r} \quad \text{….(3)}$$

If $f(x) = (x - a_r)g_r(x)$, then $f'(x) = (x - a_r)\, g_r'(x) + g_r(x)$.

Put $x = a_r$, $f'(a_r) = g_r(a_r)$.

From (3), $\frac{\phi(x)}{f(x)} = \sum_{r=1}^{n} \frac{\phi(a_r)}{g_r(a_r)} \frac{1}{x-a_r}$.

22. Let $\phi(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0$.

By the Extended Partial Fraction Theorem (see 21(b)), $\frac{\phi(x)}{f(x)} = \sum_{r=1}^{n} \frac{\phi(a_r)}{f'(a_r)} \frac{1}{x-a_r}$

$$\phi(x) = \sum_{r=1}^{n} \frac{\phi(a_r)}{f'(a_r)} \frac{f(x)}{x-a_r} = \sum_{r=1}^{n} \frac{\phi(a_r)}{f'(a_r)} (x-a_0)\ldots(x-a_{r-1})(x-a_{r+1})\ldots(x-a_n)$$

Compare coefficients of x^n term, $b_n = \sum_{r=0}^{n} \frac{\phi(a_r)}{f'(a_r)}$.

23. $$\frac{ax^2 + bx + c}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma} = \frac{A(x-\beta)(x-\gamma) + B(x-\gamma)(x-\alpha) + C(x-\alpha)(x-\beta)}{(x-\alpha)(x-\beta)(x-\gamma)}$$

$$\therefore \quad ax^2 + bx + c = A(x-\beta)(x-\gamma) + B(x-\gamma)(x-\alpha) + C(x-\alpha)(x-\beta)$$

Compare coefficients of x^2 term, $a = A + B + C$

$\therefore$ $A + B + C = 0$ iff $a = 0$ …(1)

Let $$\frac{3n+1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$$

$3n + 1 \equiv (n + 1)(n + 2) A + n(n + 2) B + C n(n + 1)$ …(2)

Put $n = 0$ in (2), $\therefore A = 1/2$.

Put $n = -2$ in (2), $\therefore C = -5/2$

From (1), $\therefore B = -A - C = -1/2 + 5/2$

$$\therefore \quad \frac{3n+1}{n(n+1)(n+2)} = \frac{\frac{1}{2}}{n} + \frac{-\frac{1}{2}+\frac{5}{2}}{n+1} + \frac{\frac{5}{2}}{n+2} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{5}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$\therefore \quad \sum_{n=1}^{N} \frac{3n+1}{n(n+1)(n+2)} = \frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{5}{2}\sum_{n=1}^{N}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{N+1}\right) + \frac{5}{2}\left(\frac{1}{2} - \frac{1}{N+2}\right) = \frac{N(7N+9)}{4(N+1)(N+2)}$$

24. **(a)** $$\frac{3+x^2}{(1-x)^2(1+x^2)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C+Dx}{1+x^2}$$

$3 + x^2 \equiv A(1 - x)(1 + x^2) + B(1 + x^2) + (C + Dx)(1 - x)^2$ ….(1)

Put $x = 1$, $4 = B(1 + 1)$, $\therefore B = 2$

Put $x = i$, $2 = (C + Dx)(1 - i)^2 = (C + Di)(-2i) = 2D - 2Ci$, $\therefore C = 0, D = 1$.

Compare the constant terms of (1), $3 = A + B + C = A + 2 + 0$, $\therefore A = 1$

$$\frac{3+x^2}{(1-x)^2(1+x^2)} = \frac{1}{1-x} + \frac{2}{(1-x)^2} + \frac{x}{1+x^2}$$

(b) **(Method 1)**

$$\frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \ldots + \frac{x^{2^{n-1}}}{1-x^{2^n}} + \ldots = \frac{x+x^2+x^3}{1-x^4} + \frac{x^4}{1-x^8} + \ldots + \frac{x^{2^{n-1}}}{1-x^{2^n}} + \ldots$$

$$= \frac{x+x^2+x^3+\ldots+x^7}{1-x^8} + \frac{x^8}{1-x^{16}} + \ldots + \frac{x^{2^{n-1}}}{1-x^{2^n}} + \ldots$$

$$= \frac{x+x^2+x^3+\ldots+x^{2^n-1}}{1-x^{2^n}} + \ldots = \lim_{n\to\infty}\frac{x+x^2+x^3+\ldots+x^{2^n-1}}{1-x^{2^n}} = \frac{\frac{x}{1-x}}{1-0} = \frac{x}{1-x}, \quad \text{if } |x| < 1.$$

(Method 2)

$$\frac{x}{1-x^2} = x(1+x^2+x^4+x^8+\ldots) = x + x^3 + x^5 + x^9 + \ldots$$

$$\frac{x^2}{1-x^4} = x^2(1+x^4+x^8+x^{16}+\ldots) = x^2 + x^6 + x^{10} + x^{17} + \ldots$$

$$\frac{x^4}{1-x^8} = x^4(1+x^8+x^{16}+x^{32}+\ldots) = x^4 + x^{12} + x^{20} + x^{36} + \ldots$$

….

Adding up all equalities, $$\frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \ldots = x + x^2 + x^3 + x^4 + \ldots = \frac{x}{1-x}$$

25. Put $u = x - 1$, $x = u + 1$

$$\frac{x^6 - x^2 + 1}{(x-1)^3} = \frac{(u+1)^6 - (u+1)^2 + 1}{u^3} = \frac{u^6 + 6u^5 + 15u^4 + 20u^3 + 14u^2 + 4u + 1}{u^3}$$

$$= u^3 + 6u^2 + 15u + 20 + \frac{14}{u} + \frac{4}{u^2} + \frac{1}{u^3} = (x-1)^3 + 6(x-1)^2 + 15(x-1) + 20 + \frac{14}{x-1} + \frac{4}{(x-1)^2} + \frac{1}{(x-1)^3}$$

$$= x^3 + 3x^2 + 6x + 10 + \frac{14}{x-1} + \frac{4}{(x-1)^2} + \frac{1}{(x-1)^3}$$

26. $\left(\frac{x}{y}+1\right)\left(\sqrt{2}-\frac{x}{y}\right)\left(\sqrt{2}-\frac{x+2y}{x+y}\right) \equiv A\left(\sqrt{2}-\frac{x}{y}\right)^2$

Put $x = 0$, $y = 1$, $1 \times \sqrt{2} \times \left(\sqrt{2} - 2\right) = A\left(\sqrt{2}\right)^2$, $\therefore\ A = 1 - \sqrt{2}$

(i) $\left(\frac{m}{n}+1\right)\left(\sqrt{2}-\frac{m}{n}\right)\left(\sqrt{2}-\frac{m+2n}{m+n}\right) \equiv \left(1-\sqrt{2}\right)\left(\sqrt{2}-\frac{m}{n}\right)^2$ .....(*)

$$\left(\sqrt{2}-\frac{m}{n}\right)\left(\sqrt{2}-\frac{m+2n}{m+n}\right) \equiv \frac{\left(1-\sqrt{2}\right)\left(\sqrt{2}-\frac{m}{n}\right)^2}{\left(\frac{m}{n}+1\right)} < 0, \quad \text{since } \left(1-\sqrt{2}\right) < 0,\ \frac{m}{n} > 0.$$

$\therefore$ $\sqrt{2}$ lies between the positive rational number $\frac{m}{n}$ and $\frac{m+2n}{m+n}$.

(ii) From (*), $\left(\frac{m}{n}+1\right)\left(\sqrt{2}-\frac{m+2n}{m+n}\right) \equiv \left(1-\sqrt{2}\right)\left(\sqrt{2}-\frac{m}{n}\right)$

$$\therefore \quad \left|\sqrt{2}-\frac{m+2n}{m+n}\right| = \left|\frac{1-\sqrt{2}}{\frac{m}{n}+1}\right|\left|\sqrt{2}-\frac{m}{n}\right|, \quad \text{and} \quad \left|\frac{1-\sqrt{2}}{\frac{m}{n}+1}\right| < \frac{1}{\left|\frac{m}{n}+1\right|} < \frac{1}{1} = 1$$

$\therefore$ $\left|\sqrt{2}-\frac{m+2n}{m+n}\right| < \left|\sqrt{2}-\frac{m}{n}\right|$ and $\therefore$ $\sqrt{2}$ is closer to $\frac{m+2n}{m+n}$ than to $\frac{m}{n}$.

27. **(a)** $(1 + x)^{n+2} = (1 + x)^2 (1 + x)^n \Leftrightarrow \sum_{r=0}^{n+2} b_r x^r = \left(1+2x+x^2\right)\sum_{r=0}^{n+2} a_r x^r$

Comparing coefficients on the term x^{r+2} on both sides, $b_{r+2} = a_r + 2a_{r+1} + a_{r+2}$ if $0 \le r \le n - 2$.

(b) Similar to no. 20.

(c) Put $n = 2$ in (b) and divide the equality by 2,

$$\frac{1}{(x+r)(x+r+1)(x+r+2)} = \frac{1}{2(x+r)} - \frac{1}{x+1} + \frac{1}{2(x+2)} \quad .....(*)$$

$$\frac{a_0}{x(x+1)(x+2)} - \frac{a_1}{(x+1)(x+2)(x+3)} + \ldots + \frac{(-1)^n a_n}{(x+n)(x+n+1)(x+n+2)} = \frac{(n+2)!}{2x(x+1)\ldots(x+n+2)}$$

$$= \sum_{r=0}^{n} \frac{(-1)^r a_r}{(x+r)(x+r+1)(x+r+2)} = \sum_{r=0}^{n} \frac{(-1)^r a_r}{2(x+r)} - \sum_{r=0}^{n} \frac{(-1)^r a_r}{x+1} + \sum_{r=0}^{n} \frac{(-1)^r a_r}{2(x+2)}$$

$$= \frac{1}{2} \times \frac{n!}{x(x+1)\ldots(x+n)} - \frac{n!}{(x+1)(x+2)\ldots(x+n+1)} + \frac{1}{2} \times \frac{n!}{(x+2)(x+3)\ldots(x+n+2)}$$

$$= \frac{n!\left[(x+n+1)(x+n+2) - 2x(x+n+2) + x(x+1)\right]}{2x(x+1)\ldots(x+n+2)} = \frac{n!(n^2+3n+2)}{2x(x+1)\ldots(x+n+2)} = \frac{(n+2)!}{2x(x+1)\ldots(x+n+2)}$$

28. **(a)** same as 21.(a),(b).

(b) $\dfrac{2x-1}{(x-1)^2}=\dfrac{2(x-1)+1}{(x-1)^2}=\dfrac{2}{x-1}+\dfrac{1}{(x-1)^2}$

(c) $B(x) = x^4 - 10x^2 + 1$, $\quad B(1) = -8$

$B'(x) = 4x^3 - 20x$ $\quad B'(1) = -16$

$B''(x) = 12x^2 - 20$ $\quad B''(1) = -8$

From (a), putting $x = 1$, $\quad \sum_{i=1}^{4}\dfrac{1}{1-x_i}=\dfrac{B'(1)}{B(1)}=\dfrac{-16}{-8}=2$

$$\sum_{i=1}^{4}\frac{1}{(1-x_i)^2}=\frac{[B'(1)]^2-B(1)B''(1)}{[B(1)]^2}=3$$

From (b), $\sum_{i=1}^{4}\dfrac{2x_i-1}{(x_i-1)^2}=\sum_{i=1}^{4}\dfrac{2}{x_i-1}+\sum_{i=1}^{4}\dfrac{1}{(x_i-1)^2}=-2\sum_{i=1}^{4}\dfrac{1}{1-x_i}+\sum_{i=1}^{4}\dfrac{1}{(1-x_i)^2}=-2(2)+3=-1$

29. $\dfrac{7-8x}{(1-x)(2-x)}=-\dfrac{1}{1-x}+\dfrac{9/2}{1-\dfrac{x}{2}}=-(1+x+x^2+\ldots+x^n+\ldots)+\dfrac{9}{2}\left(1+\dfrac{x}{2}+\dfrac{x^2}{4}+\ldots+\dfrac{x^n}{2^n}+\ldots\right)$

$=\dfrac{7}{2}+\dfrac{5}{4}x+\dfrac{1}{8}x^2-\dfrac{7}{16}x^3+\ldots+\left(\dfrac{9}{2^{n+1}}-1\right)x^n+\ldots$

The expansion is valid if $\quad |x|<1\wedge\left|\dfrac{x}{2}\right|<1$, $\quad$ i.e. $\quad |x| < 1$.

The general term $\quad T_{n+1}=\dfrac{9}{2^{n+1}}-1$. $\quad$ When $\quad n \ge 3$, $\quad T_{n+1} < 0$.

From the fourth terms onwards, the coefficients are all negative.

30. $\dfrac{1+x}{(1+2x)^2(1-x)}=\dfrac{2/9}{1-x}+\dfrac{4/9}{1+2x}+\dfrac{1/3}{(1+2x)^2}$

$\dfrac{1}{1-x}=1+x+x^2+\ldots+x^n+\ldots$

$\dfrac{1}{1+2x}=1-2x+(2x)^2+\ldots+(-1)^n(2x)^n+\ldots$

$\dfrac{1}{(1+2x)^2}=1-2(2x)+3(2x)^2+\ldots+(-1)^n(n+1)(2x)^n+\ldots$

The above expansions are valid if $\quad |x|<1\wedge|2x|<1$, $\quad$ i.e. $\quad |x|<\dfrac{1}{2}$.

The coefficient of x^n term = $\dfrac{2}{9}+\dfrac{4}{9}(-1)^n 2^n+\dfrac{1}{3}(-1)^n(n+1)2^n=\dfrac{2}{9}+\dfrac{1}{9}(-1)^n 2^n(3n+7)$.

31. $\dfrac{2}{(1-2x)^2(1+4x^2)}=\dfrac{1}{1-2x}+\dfrac{1}{(1-2x)^2}+\dfrac{2x}{1+4x^2}$

$=\left[1+2x+(2x)^2+\ldots+(2x)^{4n1}+(2x)^{4n+1}+\ldots\right]$

$+\left[1+2\ (2x)+3(2x)^2+\ldots+(2n+1)(2x)^n+\ldots+(4n+1)(2x)^{4n}+(4n+2)(2x)^{4n+1}+\ldots\right]$

$+\left[2x-(2x)(4x^2)+\ldots-(2x)(4x^2)^{2n-1}+(2x)(4x^2)^{2n}-\ldots\right]$

The coefficient of $x^{4n} = 2^{4n} + (4n + 1)2^{4n} = (2n + 1)\, 2^{4n+1}$

The coefficient of $x^{4n+1} = 2^{4n+1} + (4n + 2)2^{4n+1} + 2 \times 4^{2n} = (n + 1)\, 2^{4n+3}$.

The above expansions are valid if

$$|2x| < 1 \wedge |4x^2| < 1 \Leftrightarrow |x| < \frac{1}{2} \wedge (2x-1)(2x+1) < 0 \Leftrightarrow |x| < \frac{1}{2} \wedge -\frac{1}{2} < x < \frac{1}{2} \Leftrightarrow |x| < \frac{1}{2}$$

32. $$\frac{x^2+1}{(x-3)^2(x-2)} = \frac{5}{x-2} - \frac{4}{x-3} + \frac{10}{(x-3)^2}$$

$$\frac{1}{x-2} = \frac{1}{2}\left(\frac{1}{1-x/2}\right) = -\frac{1}{2}\left[1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \ldots + \left(\frac{x}{2}\right)^n + \ldots\right]$$

$$\frac{1}{x-3} = \frac{1}{3}\left(\frac{1}{1-x/3}\right) = -\frac{1}{3}\left[1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \ldots + \left(\frac{x}{3}\right)^n + \ldots\right]$$

$$\frac{1}{(x-3)^2} = \left(\frac{1}{3}\right)^2\left(\frac{1}{1-x/3}\right)^2 = \left(-\frac{1}{3}\right)^2\left[1 + 2\left(\frac{x}{3}\right) + 3\left(\frac{x}{3}\right)^2 + \ldots + (n+1)\left(\frac{x}{3}\right)^n + \ldots\right]$$

The coefficient of x^n term = $5 \times \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^n + (-4)\left(-\frac{1}{3}\right)\left(\frac{1}{3}\right)^n + 10\left(-\frac{1}{3}\right)^2 (n+1)\left(\frac{1}{3}\right)^n = \frac{10n+22}{3^{n+2}} - \frac{5}{2^{n+1}}$.

The above expansions are valid if $\left|\frac{x}{3}\right| < 1 \wedge \left|\frac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$.

33. $$\frac{2+x^2}{(2-x)^2(4+x)} = -\frac{1/2}{2-x} + \frac{1}{(2-x)^2} + \frac{1/2}{4+x} = \frac{-1/4}{1-\frac{x}{2}} + \frac{1/4}{\left(1-\frac{x}{2}\right)^2} + \frac{1/8}{1+\frac{x}{4}}$$

$$= -\frac{1}{4}\left[1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \ldots + \left(\frac{x}{2}\right)^n + \ldots\right] + \frac{1}{4}\left[1 + 2\left(\frac{x}{2}\right) + 3\left(\frac{x}{2}\right)^2 + \ldots + (n+1)\left(\frac{x}{2}\right)^n\right]$$
$$+ \frac{1}{8}\left[1 - \frac{x}{4} + \left(\frac{x}{4}\right)^2 + \ldots + (-1)^n\left(\frac{x}{4}\right)^n + \ldots\right]$$

The coefficient of x^n term = $\frac{n}{2^{n+2}} + (-1)^n \frac{1}{2^{2n+3}}$.

The above expansions are valid if $|x| < 2$.

34. $$\frac{3x+4}{(x+1)(x+2)^2} = \frac{1}{1+x} - \frac{1}{2+x} + \frac{2}{(2+x)^2} = \frac{1}{1+x} - \frac{1}{2}\frac{1}{1+\frac{x}{2}} + \frac{1}{2}\frac{1}{\left(1+\frac{x}{2}\right)^2}$$

$$= \left[1 - x + x^2 - x^3 + \ldots\right] - \frac{1}{2}\left[1 - \frac{x}{2} + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \ldots\right] + \frac{1}{2}\left[1 - 2\left(\frac{x}{2}\right) + 3\left(\frac{x}{2}\right)^2 - 4\left(\frac{x}{2}\right)^3 + \ldots\right]$$

$$= 1 - \frac{5}{4}x + \frac{5}{4}x^2 - \frac{19}{16}x^3 + \ldots$$

The restriction on the values is $|x| < 1 \wedge \left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 1$.